11. The Ternary System

Somewhere in the darkness a woman sang in a high wild voice and the tune had no start and no finish and was made up of only three notes which went on and on and on.

Carson McCullers,
The Ballad of the Sad Café

NOW AND THEN a cultural anthropologist, eager to push mathematics into the folkways, will point to the use of different number systems in primitive societies as evidence that laws of arithmetic vary from culture to culture. But of course the same old arithmetic is behind every number system. The systems are nothing more than different languages: different ways of uttering, symbolizing, and manipulating the *same* numbers. Two plus two is invariably four in any notation, and it is always possible to translate perfectly from one number language to another.

Any integer except 0 can furnish the base, or radix, of a number system. The simplest notation, based on 1, has only one symbol: the notches an outlaw cuts in his gun or the beads a billiard player slides along a wire to record his score. The binary system has two symbols: 0 and 1. The decimal system, now universal throughout the civilized world, uses ten symbols. The larger the base, the more compactly a large number can be written. The decimal number 1,000 requires ten digits in binary notation (1111101000) and 1,000 digits in the 1-system. On the debit side, a large base means more digits to memorize and larger tables of addition and multiplication.

From time to time reform groups, fired with almost religious zeal, seek to overthrow what has been called the "tyranny of 10" and replace it with what they believe to be a more efficient radix. In recent years the duodecimal system, based on 12, has been the most popular. Its chief advantage is that all multiples of the base can be evenly halved, thirded, and quartered. (The unending decimal fraction .3333 . . . , which stands for 1/3, becomes a simple .4 in the 12-system.) There have been advocates of a 12-base since the sixteenth century, including such personages as Herbert Spencer, John Quincy Adams, and George Bernard Shaw. H. G. Wells has the system adopted before the year 2100 in his novel *When the Sleeper Wakes.* There is even a Duodecimal Society of America. (Its headquarters are at 20 Carleton Place, Staten Island, New York 10304.) It publishes *The Duodecimal Bulletin* and *Manual of the Dozen System* and supplies its "dozeners" with a slide rule based on a radix of 12. The society uses an X symbol (called dek) for 10 and an inverted 3 (called el) for 11. The first three powers of 12 are do, gro, mo; thus the duodecimal number 111X is called mo gro do dek.

Advocates of radix 16 have produced the funniest literature. In 1862 John W. Nystrom published privately in Philadelphia his *Project of a New System of Arithmetic, Weight, Measure, and Coins, Proposed to Be Called the Tonal System, with Sixteen to the Base.* Nystrom urges that numbers 1 through 16 be called an, de, ti, go, su, by, ra, me, ni, ko, hu, vy, la, po, fy, ton. Joseph Bowden, who was a mathematician at Adelphi College, also considered 16 the best radix but preferred to keep the familiar names for numbers 1 through 12, then continue with thrun, fron, feen, wunty. In Bowden's notation 255 is written ƨƨ and pronounced "feenty feen." (See Chapter 2 of his *Special Topics in Theoretical Arithmetic*, privately published; Garden City, New York: 1936.)

It seems unlikely that the "tyranny of 10" will soon be toppled, but that does not prevent the mathematician from using whatever number system he finds most useful for a given task. If a structure is rich in two values, such as the on-off values of computer circuits, the binary system may be much more efficient than the decimal system. Similarly, the ternary, or 3-base, system is often the most efficient way to analyze structures rich in three values. In the quotation that opens this chapter Carson McCullers is writing about herself. She is the woman singing in the darkness about that grotesque triangle in which Macy loves Miss Amelia, who loves Cousin Lymon, who loves Macy. To a mathematician this sad, endless round of unrequited love suggests the endless round of a base-3 arithmetic: each note ahead of another, like the numbers on an eternally running three-hour clock.

In ternary arithmetic the three notes are 0, 1, 2. As you move left along a ternary number, each digit stands for a multiple of a higher power of 3. In the ternary number 102, for example, the 2 stands for 2×1. The 0 is a "place holder," telling us that no multiples of 3 are indicated. The 1 stands for 1×9. We sum these values, $2 + 0 + 9$, to obtain 11, the decimal equivalent of the ternary number 102. Figure 80 shows the ternary equivalents of the decimal numbers 1 through 27. (A Chinese abacus, by the

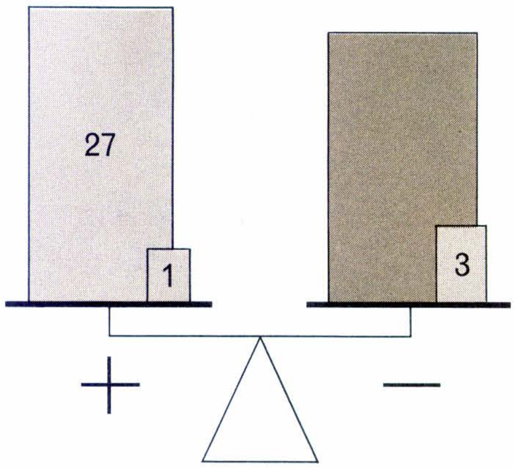
DECIMAL NUMBERS	TERNARY NUMBERS			
	3^3	3^2	3^1	3^0
1				1
2				2
3			1	0
4			1	1
5			1	2
6			2	0
7			2	1
8			2	2
9		1	0	0
10		1	0	1
11		1	0	2
12		1	1	0
13		1	1	1
14		1	1	2
15		1	2	0
16		1	2	1
17		1	2	2
18		2	0	0
19		2	0	1
20		2	0	2
21		2	1	0
22		2	1	1
23		2	1	2
24		2	2	0
25		2	2	1
26		2	2	2
27	1,	0	0	0

80. Ternary numbers 1 through 27

way, is easily adapted to the ternary system. Just turn it upside down and use the two-bead section.)

Perhaps the most common situation lending itself to ternary analysis is provided by the three values of a balance scale: either one pan goes down or the other pan goes down, or the pans balance. As far back as 1624, in the second edition of a book on recreational mathematics, Claude Gasper Bachet asked for the smallest number of weights needed for weighing any object with an integral weight of from 1 through 40 pounds. If the weights are restricted to one side of the scale, the answer is six: 1, 2, 4, 8, 16, 32 (successive powers of 2). If the weights may go on either pan, only four are needed: 1, 3, 9, 27 (successive powers of 3).

To determine how weights are placed to weigh an object of n pounds, we first write n in the ternary system. Next we change the form of the ternary number so that instead of expressing its value with the symbols 0, 1, 2 we use the symbols, 0, 1, −1. To do this each 2 is changed to −1, then the digit to the immediate left is increased by 1. If this produces a new 2, it is eliminated in the same way. If the procedure creates a 3, we replace the 3 with 0 and add 1 to the left. For instance, suppose the weight is 25 pounds, or 221 in ternary notation. The first 2 is changed to −1, then 1 is added to the left, forming the number 1 −1 2 1. The remaining 2 is now changed to −1, and 1 is added to the left, making the number 1 0 −1 1. This new ternary number is equivalent to the old one ($27 + 0 - 3 + 1 = 25$), but now it is in a form that tells us how to place the weights. Plus digits indicate weights that go in one pan, minus digits indicate weights that go in the other pan. The object to be weighed is placed on the minus side. Figure 81 shows how the three weights are



81. *How to weigh a 25-pound object*

placed for weighing a 25-pound object.

The base-3 system using the symbols −1, 0, +1 is called the "balanced ternary system." A good discussion of it can be found in Donald E. Knuth's *Seminumerical Algorithms* (New York: Addison-Wesley,1969; pages 173–175). "So far no substantial application of balanced ternary notation has been made," Knuth concludes, "but perhaps its symmetric properties and simple arithmetic will prove to be quite important some day (when the 'flip-flop' is replaced by a 'flip-flap-flop')."

Suppose you wish to determine the weight of a single object known to have an integral weight of from 1 through 27 pounds. What is the smallest number of weights needed, assuming that they may be placed on either pan? There is no catch, but the question is tricky and the answer is not what you are first likely to think.

A more sophisticated balance-scale problem (dozens of papers have discussed it since it first sprang up, seemingly out of nowhere, in 1945) is the problem of the 12 coins. They are exactly alike except for one counterfeit, which weighs a bit more or a bit less than the others. With a balance scale and *no* weights, is it possible to identify the counterfeit in three weighings and also know if it is underweight or overweight?

Although I constantly receive letters asking about this problem, I have avoided writing about it because it was so ably discussed by C. L. Stong in "The Amateur Scientist" column of *Scientific American* for May, 1955. Now we shall see how one solution (there are many others) is linked with the ternary system.

First, list the ternary numbers from 1 through 12. To the right of each number write a second ternary number obtained from the first by changing each 0 to 2, each 2 to 0 [*see Figure* 82]. Next, find every num-

82. *Ternary numbers for 12-coin problem*

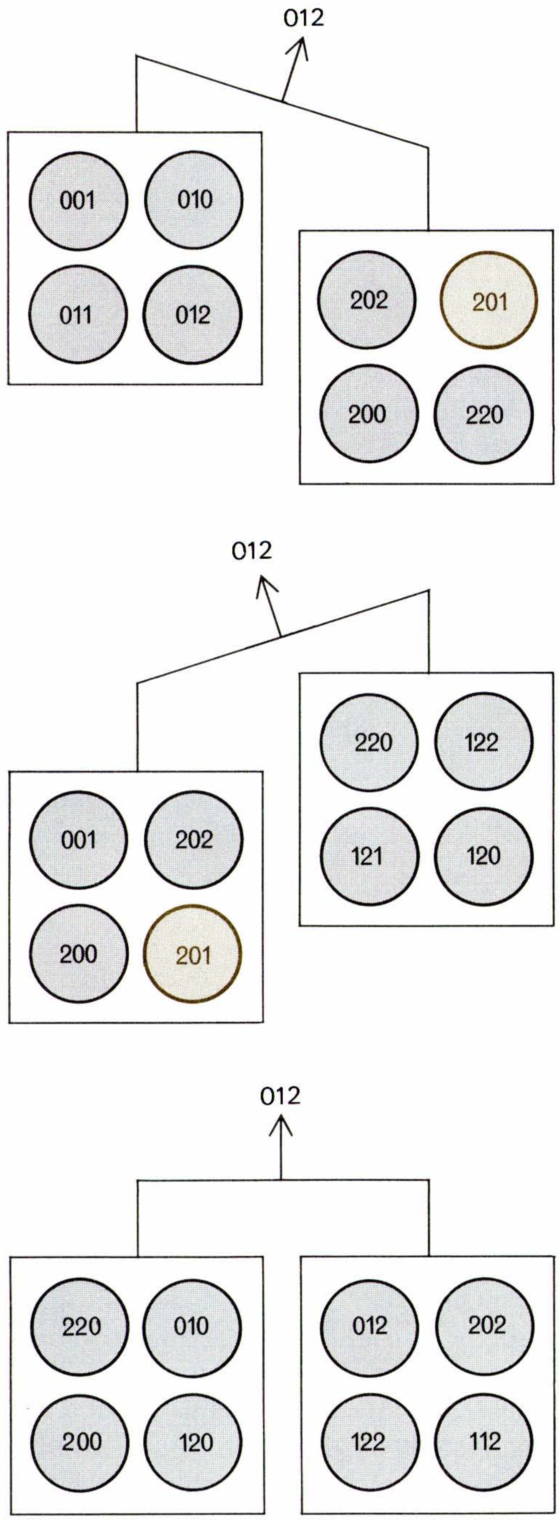
1	001	221
2	002	220
3	010	212
4	011	211
5	012	210
6	020	202
7	021	201
8	022	200
9	100	122
10	101	121
11	102	120
12	110	112

ber that contains as the first *unlike* digits one of the following pairs of adjacent digits: 01, 12, 20. Assign one of these 12 numbers (shown in color) to each of the 12 coins.

For the first weighing the four coins with a first digit of 0 go left, the four with a first digit of 2 go right. If the pans balance, put down 1 as the first digit of the counterfeit. If the left pan goes down, the counterfeit's first digit is 0; if the right pan goes down, it is 2.

For the second weighing the four coins with a middle digit of 0 go left, the four with a middle digit of 2 go right. The same procedure is followed to obtain the middle digit of the counterfeit. On the third weighing, coins with final digits of 0 go left, those with final digits of 2 go right, and the last digit of the counterfeit is obtained as before. Figure 83 shows the three weighings that identify the counterfeit as coin 201. When the coin is overweight, as in this case, the number given by the three weighings is the actual number of the coin. If the three weighings give a number *not* assigned to a coin, then the coin is *underweight*. Its number is obtained by substituting a 0 for each 2, and a 2 for each 0.

Scores of simplified versions of this procedure have been devised. The best I know comes from W. Fitch Cheney, Jr., a mathematician at the University of Hartford. Label the coins with the letters of SILENT COWARD. The three weighings are SCAN



83. Three weighings to identify a counterfeit coin against WORD, SCAR against LINE, SLOT against RAID. Put a ring around each word that goes down. If a pair balances, mark out all its letters from all six words. Inspect the circled words. If there is a letter not crossed out that appears in each word, it indicates the false coin and the coin is overweight. If there is no such letter, you are sure to find one not crossed out in each of the uncircled words. It then indicates an underweight counterfeit. Other key words can, of course, be devised. L. E. Card, intrigued by Cheney's SILENT COWARD, found two dozen sets, of which I cite only one: CRAZY WEIGHTS: CITY-HAZE, GREW-HAZY, AND WISH-TRAY.

The problem has been generalized. In four weighings one can identify the false coin, and tell whether it is light or heavy, among a maximum of $3^1+3^2+3^3=39$ coins; five weighings will take care of $3^1+3^2+3^3+3^4=120$ coins, and so on. More compactly, n weighings take care of $\frac{1}{2}(3^n-3)$ coins. It is worth noting that a counterfeit among 13 coins can be found in three weighings if one need not know whether it is heavier or lighter (simply put the 13th coin aside and if you fail to find the counterfeit among the 12, the 13th coin is it); to know whether the false coin is heavier or lighter, three weighings also suffice for 13 coins if you add a 14th coin known to be genuine.

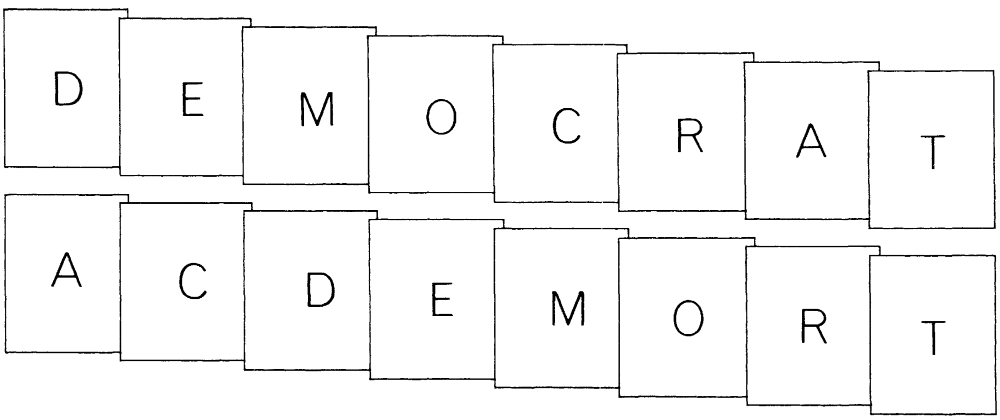
Many card tricks are closely related to the 12-coin problem. One of the best is known as Gergonne's three-pile problem after the French mathematician Joseph Diez Gergonne, who first studied it early in the nineteenth century. Someone is asked to look through a packet of 27 cards and fix one in his mind. He holds the packet face down, deals the cards face up into a row of three, then continues dealing on top of these cards, left to right, until all 27 have been dealt into three face-up piles of nine cards each. After telling the magician which pile contains his chosen card, he assembles the piles by placing them on top of one another, in any order he wishes, turns the packet face down and again deals them into three face-up piles. Once more he indicates the pile in which his card fell. This is repeated a third time, then the assembled packet is placed face down on the table. The magician, who has not touched the cards throughout the entire procedure, names the position of the chosen card.

The secret lies in observing, at each pickup, whether the pile with the selected card goes on the top, the bottom, or in the middle of the assembled facedown packet. These positions are designated 0 for the top, 1 for the middle, 2 for the bottom. The ternary number expressed by the three pickups, written from *right to left*, is the number of cards above the chosen card after the final pickup. For example, suppose the first pickup puts the pile on the top (0), the second on the bottom (2), the third in the middle (1). These digits, written right to left, give the ternary number 120, or 15 in the decimal scale. Fifteen cards are therefore above the selected one, making it the 16th card from the top. Of course, the trick can be done just as easily in reverse. The spectator chooses any number from 1 through 27, then the magician, making the pickups himself, brings the card to that number from the top.

If in dealing into three piles one is permitted to put each card on *any* pile, a powerful sorting method results. At this point the reader is asked to obtain eight file cards and print on each card one of the letters in the word DEMOCRAT. Arrange the cards into a packet, letter sides down, that spells DEMOCRAT from the top down [*see top illustration of Figure 84*]. You wish to rearrange the cards so that, from the top down, they are in alphabetical order as shown in Figure 84, bottom. It is easily done in one deal. Turn the top card, *D*, face up and place it as the first card of pile 1. The next three cards, *E, M, O,* go on top of the *D*. *C* becomes the first card of pile 2, *R* goes back on pile 1, *A* starts pile 3, and *T* goes on pile 1. Assemble by putting pile 1 on 2 and those cards on 3; then turn the packet face down. You will find the cards in alphabetical order, top to bottom. A single deal is also sufficient, as you can easily discover, for changing the alphabetized order back to DEMOCRAT.

Put the DEMOCRAT cards aside and make a new set that spells REPUBLICAN. Can *this* set be alphabetized in one operation? No, it cannot. What is the smallest number of operations necessary? Remember, the initial packet of face-down cards must spell the word from the top down. Each card is dealt face up, the piles are picked up in any order, then the packet is turned face down to conclude one operation. After the last operation the cards must be in the order ABCEILNPRU, top to bottom. If you solve this problem, see if you can determine the minimum number of operations needed to change the order back to REPUBLICAN. And if both problems seem too easy, try a set of cards that spell SCIENTIFIC AMERICAN. In

84. Original (top) *and desired sorting of DEMOCRAT cards*



the answer section I explain how all sorting problems of this type can be solved quickly by a simple application of ternary numbers, and I also answer the problem of the weights.

Answers

The minimum number of weights needed to weigh 27 boxes with integral weights of from 1 through 27 pounds, assuming that weights may be placed on either side of a balance scale, is three: 2, 6, and 18 pounds. (They represent doublings of successive powers of 3.) These weights will achieve an exact balance for every even number of pounds from 1 through 27. The odd weights are determined by checking the even weights directly above and below; for example, a box of 17 pounds is identified by the fact that it weighs less than 18 and more than 16 pounds. (Mitchell Weiss of Downey, California, provided this pleasant twist on an old problem.)

The task of alphabetizing the letters of REPUBLICAN by dealing letter cards into three piles can be solved in two operations. First, write down the letters in alphabetical order: ABCEILNPRU. *A* is the first letter, so we place a *0* above the letter *A* in the word REPUBLICAN. We move *right* along the word in search of *B*, the second letter, but we do not find it. Because we are forced to move *left* to reach *B*, we put *1* above it. We continue to move right in search of *C*. This time we find it on the right, so we label it with *1* also. The next letter, *E*, forces us to move left again, therefore we label it *2*. *I* is to the right of *E*, so it gets *2* also, but *L* carries us left again, so it gets *3*. In short, we raise the number only when we have to move left to find the letter. This is how the final result appears:

5	2	4	5	1	3	2	1	0	3
R	E	P	U	B	L	I	C	A	N

On each letter card write the ternary equivalent of the decimal number assigned to that letter. The cards are held in a face-down packet, spelling REPUBLICAN from the top down. Imagine that the three piles are numbered, left to right, 0, 1, 2. Turn over the top card, *R*. Its ternary number is 12. The *last* digit, 2, tells you to deal the card to pile 2 (the end pile on the right). The next card, *E*, has a ternary number of 02; it also goes on the right end pile. Continue in this way, dealing each card to the pile indicated by the final digit. The piles are always assembled from right to left by putting the last pile (2) on the center pile (1), then all those cards on the first pile (0). Turn the packet face down and deal once more, this time dealing as indicated by the *first* digits of each ternary number. Assemble as before. The cards are now alphabetized.

To put the cards back in their original order a new analysis of the letters must be made, assigning them a new set of numbers:

5	2	4	1	3	2	5	1	0	1
A	B	C	E	I	L	N	P	R	U

Two operations will return the cards to

their initial order, but the sorting procedure is not the same as before. If the decimal numbers assigned to the letters go above 8, then a ternary number for a letter will require more than two digits, and the number of required operations will be more than two. It is easy to see that the minimum number of operations is given by the number of digits in the highest ternary number. To alphabetize SCIENTIFIC AMERICAN the letters are numbered:

6	1	4	2	5	6	4	3	3	1
S	C	I	E	N	T	I	F	I	C
	0	4	2	5	3	1	0	4	
	A	M	E	R	I	C	A	N	

Because the highest number, 6, has only two digits in its ternary form, only two operations are called for. However, to reverse the procedure, changing the alphabetized order back to SCIENTIFIC AMERICAN, the highest number is 10. This has three ternary digits, therefore three operations are necessary. If the reader will test the system on longer phrases or sentences, he will be astonished at how few operations are required for what seems to be an enormously difficult sorting job. One can generalize the method to any number of piles, n, simply by writing numbers in a system based on n.

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